On the algebraic K-theory of the coordinate axes over the integers

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Abstract

We show that $K_{2i}(\mathbb{Z}[x,y]/(xy),(x,y))$ is free abelian of rank 1 and that $K_{2i+1}(\mathbb{Z}[x,y]/(xy),(x,y))$ is finite of order $(i!)^2$. We also compute $K_{2i+1}(\mathbb{Z}[x,y]/(xy),(x,y))$ in low degrees.

1 Introduction

Let k be a ring, let $A = k[x,y]/(xy)$, and let $I = (x,y)$ be the augmentation ideal. Then the K-groups of A can be expressed as a direct sum

$$K_q(A) = K_q(k) \oplus K_q(A,I).$$

In [7], Hesselholt evaluated the algebraic K-groups $K_*(A,I)$ when k is a regular $\mathbb{F}_p$-algebra in terms of the big de Rham-Witt forms of k. Here we study the case where $k = \mathbb{Z}$ is the ring of integers. It was proved by Geller, Reid, and Weibel [6] that for every nonnegative integer q the abelian group $K_q(\mathbb{Z}[x,y]/(xy),(x,y))$ has rank 0 if q is odd and 1 if q is even. We prove the following result:

Theorem A. *For any $i \geq 0$*

1. *The abelian group $K_{2i}(\mathbb{Z}[x,y]/(xy),(x,y))$ is free of rank 1.*
2. *The abelian group $K_{2i+1}(\mathbb{Z}[x,y]/(xy),(x,y))$ is finite of order $(i!)^2$.*

We currently do not know the group structure of the finite abelian group in degree $2i+1$, except for small values of i. Note that this result agrees with the calculation by Dennis and Krusemeyer [4] of the group in degree 2.

To prove Theorem A, we use the cyclotomic trace map [3] to relate the K-groups in question to certain birelative topological cyclic homology groups. Work of Hesselholt [7] gives a formula for these topological cyclic homology groups in terms of $RO(S^1)$-graded TR-groups of $\mathbb{Z}$. In particular, the topological

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cyclic homology groups in question can be expressed in terms of equivariant homotopy groups of the form

$$\mathrm{TR}^r_{q-\lambda}(\mathbb{Z}) = \pi_{q-\lambda} T(\mathbb{Z})^{C_r} = [S^q \wedge (S^1/C_r)_+, S^\lambda \wedge T(\mathbb{Z})]_{S^1}$$

where $T(\mathbb{Z})$ denotes the topological Hochschild S^1-spectrum of $\mathbb{Z}$, λ is a finite dimensional real representation of S^1 and $C_r \subset S^1$ is the cyclic group of order r. Using this relationship the K-groups we are studying can be written in terms of these $RO(S^1)$-graded TR-groups of $\mathbb{Z}$. This relationship is recalled in Section 2.

The proof of the main theorem is then reduced to computing TR-groups of the form $\mathrm{TR}^r_{q-\lambda}(\mathbb{Z})$. In Section 3 we make the computations necessary to finish the proof. To make these computations we rely on earlier work by the authors [1] and also joint work with Hesselholt [2].

Although we are unable to determine the group structure of the odd K-groups in general, in Section 4 we compute the odd K-groups in some low degrees.

2 The relationship with $RO(S^1)$-graded TR groups

Given a pointed monoid Π, topological Hochschild homology of the pointed monoid algebra $k(\Pi)$ splits S^1-equivariantly as

$$T(k(\Pi)) \simeq T(k) \wedge B^{cy}(\Pi),$$

where $B^{cy}(\Pi)$ denotes the cyclic bar construction on Π.

Following Hesselholt [7], we define $\Pi^2 = \{0, 1, x, x^2, \ldots, y, y^2, \ldots\}$, so $k(\Pi^2) \cong k[x,y]/(xy)$. The main part of the proof in [7] is an analysis of the S^1-equivariant homotopy type of $B^{cy}(\Pi^2)$. This leads to a formula for $\mathrm{TC}(A, I)$ in terms of the $RO(S^1)$-graded TR groups of k which is valid even if k is not a regular $\mathbb{F}_p$-algebra.

Let $B = k[x] \times k[y]$ be the normalization of A, and let $K(A, B, I)$ be the iterated mapping fiber of the diagram

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & K(A/I) \arrow[d] \\ K(B) \arrow[r] & K(B/I) \end{tikzcd}$$

For a regular ring k, $K_*(A, I)$ is isomorphic to $K_*(A, B, I)$. By a result of Geisser and Hesselholt [5], the cyclotomic trace map induces an isomorphism

$$K_q(A, B, I; \mathbb{Z}/p^v) \to TC_q(A, B, I; p, \mathbb{Z}/p^v)$$

for each prime p and any $v \geq 1$. So, we focus our attention on computing $TC_q(A, B, I; p, \mathbb{Z}/p^v)$.

To compute topological cylic homology, we must first understand topological Hochschild homology. Hesselholt [7] showed that the birelative topological Hochschild spectrum splits as

$$T(A,B,I) \simeq \bigvee_{i\geq 1} T(k) \wedge S^{\lambda_i} \wedge \Sigma^{-1}(S^1/C_i)_+,$$

where S^{λ_i} is the 1-point compactification of the real S^1-representation

$$\lambda_i = \mathbb{C}(1) \oplus \ldots \oplus \mathbb{C}(i).$$

Here $\mathbb{C}(i)$ denotes the one-dimensional complex S^1-representation defined by $\mathbb{C}(i) = \mathbb{C}$ with S^1 acting from the left by $z \cdot w = z^i w$. Recall that S^1 acts on birelative topological Hochschild homology. We write $\mathrm{TR}^n(A,B,I;p) = T(A,B,I)^{C_{p^{n-1}}}$ for the fixed point spectrum under the action of $C_{p^{n-1}} \subset S^1$, the cyclic group of order p^{n-1}. These TR-spectra are connected by maps R, F, and V, and a homotopy limit over $R, F : \mathrm{TR}^n(A,B,I;p) \to \mathrm{TR}^{n-1}(A,B,I;p)$ gives us the birelative topological cyclic homology spectrum $\mathrm{TC}(A,B,I;p)$.

The splitting above then leads to the following formula [7]:

$$\mathrm{TC}_q(A,B,I;p) \cong \prod_{p \not| d} \lim_R \mathrm{TR}^r_{q-\lambda_{p^{r-1}d}}(k;p).$$

When $k = \mathbb{Z}$, the K-groups of A are finitely generated, so to prove the main theorem it suffices to show that $\mathrm{TC}_{2i}(A,B,I;p) \cong \mathbb{Z}$ and $\mathrm{TC}_{2i+1}(A,B,I;p)_{(p)}$ has order the p-primary component of $(i!)^2$ for each prime p.

3 Proof of Theorem A

Theorem A follows from the following two results:

Proposition 3.1. *The abelian group* $\lim_R \mathrm{TR}^r_{2i-\lambda_{p^{r-1}d}}(\mathbb{Z};p)$*, for* p *not dividing* d*, is free of rank* 1 *if* $i = p^s d$ *for some* $s \geq 0$ *and zero otherwise.*

Proof. By [2, Theorem B], each $\mathrm{TR}^r_{2i-\lambda_{p^{r-1}d}}(\mathbb{Z};p)$ is torsion free of rank equal to the number of integers $0 \leq s < r$ such that $i = \dim_{\mathbb{C}}(\lambda^{C_{p^s}}_{p^{r-1}d})$. The connecting map R in the limit system is an isomorphism for r sufficiently large, so the rank of $\lim_R \mathrm{TR}^r_{2i-\lambda_{p^{r-1}d}}(\mathbb{Z};p)$ is equal to the number of integers s such that $i = \dim_{\mathbb{C}}(\lambda^{C_{p^{r-1-s}}}_{p^{r-1}d})$. Using that $\lambda^{C_{p^{r-1-s}}}_{p^{r-1}d} = \lambda_{p^s d}$ the result follows. □

Corollary 3.2. *The abelian group*

$$\prod_{p \not| d} \lim_R \mathrm{TR}^r_{2i-\lambda_{p^{r-1}d}}(\mathbb{Z};p)$$

is free of rank 1.

Proposition 3.3. *The abelian group*

$$\prod_{p \nmid d} \lim_R \mathrm{TR}^r_{2i+1-\lambda_{p^{r-1}d}}(\mathbb{Z};p)_{(p)}$$

is finite of order the p-primary part of $(i!)^2$.

Proof. By [2, Theorem B], the inverse limit $\lim_R \mathrm{TR}^r_{2i+1-\lambda_{p^{r-1}d}}(\mathbb{Z};p)$ is isomorphic to $\mathrm{TR}^r_{2i+1-\lambda_{p^{r-1}d}}(\mathbb{Z};p)$ for the unique r with $p^{r-1}d \leq i < p^r d$.

Using *loc. cit.* and induction we find that

$$\begin{aligned}|\mathrm{TR}^r_{2i+1-\lambda_{p^{r-1}d}}(\mathbb{Z};p)| &= \prod_{0\leq k\leq r-1} p^{r-1-k}(i+1-\dim_{\mathbb{C}}(\lambda^{C_{p^k}}_{p^{r-1}d})) \\ &= \prod_{0\leq k\leq r-1} p^{r-1-k} \cdot \prod_{0\leq k\leq r-1} (i+1-p^{r-1-k}d).\end{aligned}$$

Taking the product over all d not divisible by p we see that we get a contribution of $p^{\nu_p(j)}$ for each $1 \leq j \leq i$ from the first product, which gives us one factor of the p-primary part of $i!$, and a factor of $i+1-j$ for each $1 \leq j \leq i$ from the second product, which gives us the other factor of $i!$. $\square$

This finishes the proof of Theorem A.

4 Low-dimensional calculations

Now we turn to explicit computations of the K-groups in low degrees. We find the following:

Theorem 4.1. *Let $A = \mathbb{Z}[x,y]/(xy)$ and let $I = (x,y)$. Then*

$$\begin{aligned}K_1(A,I) &= 0 \\ K_3(A,I) &= 0 \\ K_5(A,I) &\cong \mathbb{Z}/4 \\ K_7(A,I) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/9 \\ K_{11}(A,I) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/25\end{aligned}$$

Proof. It follows immediately from Theorem A that $K_1(A,I) = K_3(A,I) = 0$. For the rest of the proof we will freely use [1, Theorem 1.1], which tells us the number of summands of each TR-group, and [2, Theorem B], which tells us the order of each TR-group.

We find that $K_5(A,I) \cong \mathrm{TR}^2_{5-\lambda_2}(\mathbb{Z};2)$, which has order 4 and one summand. Hence $K_5(A,I)$ is as claimed. Similarly,

$$K_7(A,I) \cong \mathrm{TR}^2_{7-\lambda_2}(\mathbb{Z};2) \oplus \mathrm{TR}^2_{7-\lambda_3}(\mathbb{Z};3).$$

The 2-primary part has order 4 and two summands, and the 3-primary part has order 9 and one summand. Hence $K_7(A, I)$ is as claimed.

For $K_{11}(A, I)$, we find that

$$K_{11}(A, I) \cong \mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2) \oplus \mathrm{TR}^2_{11-\lambda_3}(\mathbb{Z}; 3) \oplus \mathrm{TR}^2_{11-\lambda_5}(\mathbb{Z}; 5).$$

We find that $\mathrm{TR}^2_{11-\lambda_3}(\mathbb{Z}; 3) \cong \mathbb{Z}/9$ and $\mathrm{TR}^2_{11-\lambda_5}(\mathbb{Z}; 5) \cong \mathbb{Z}/25$, while $\mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2)$ is identified in Propositiion 4.2 below.

□

To compute $\mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2)$, we will use the Tate spectral sequence. Recall that we have the following fundamental diagram of horizontal long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{q-\lambda} T_{hC_{p^n}} \arrow[r, "N"] \arrow[d, "="] & \mathrm{TR}^{n+1}_{q-\lambda} \arrow[r, "R"] \arrow[d, "\Gamma_n"] & \mathrm{TR}^n_{q-\lambda'} \arrow[r] \arrow[d, "\hat{\Gamma}_n"] & \cdots \\
\cdots \arrow[r] & \pi_{q-\lambda} T_{hC_{p^n}} \arrow[r, "N^h"] & \pi_{q-\lambda} T^{hC_{p^n}} \arrow[r, "R^h"] & \pi_{q-\lambda} T^{tC_{p^n}} \arrow[r] & \cdots
\end{tikzcd}$$

Here $T = T(\mathbb{Z})$ is the topological Hochschild spectrum, $T^{tC_{p^n}}$ is the Tate spectrum, $T^{hC_{p^n}}$ is the homotopy fixed point spectrum, $T_{hC_{p^n}}$ is the homotopy orbit spectrum, and $\mathrm{TR}^n = \mathrm{TR}^n(\mathbb{Z}; p)$. As in [1], $\lambda' = \rho_p^*(\lambda^{C_p})$ where $\rho_p : S^1 \to S^1/C_p \cong S^1$ is the p'th root map. By Tsalidis' Theorem [9] extended to the $RO(S^1)$-graded context, the map Γ_n is an isomorphism for $q > 2\dim_{\mathbb{C}}(\lambda)$ and $\hat{\Gamma}_n$ is an isomorphism for $q > 2\dim_{\mathbb{C}}(\lambda')$.

The Tate spectral sequence, which computes $\pi_{*-\lambda} T^{tC_{p^n}}$, has E_2 term

$$\hat{E}_2^{s,t} = \widehat{H}^s(C_{p^n}; \pi_{t-\lambda} T(\mathbb{Z})).$$

Here $\pi_{t-\lambda} T(\mathbb{Z}) \cong \pi_{t-2\dim_{\mathbb{C}}(\lambda)} T(\mathbb{Z})$. By restricting to the second quadrant, we get a corresponding spectral sequence which computes $\pi_{*-\lambda} T^{hC_{p^n}}$. These spectral sequences were studied in detail with mod p coefficients in [1]. While understanding the Tate spectral sequence with integral coefficients remains an extremely difficult problem, an essential ingredient in the proof of [2, Theorem B] is that all non-zero differentials go from even to odd total degree. A partial understanding of the C_4-Tate spectral sequence with mod 4 coefficients will be enough to compute $\mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2)$.

Proposition 4.2. *We have*

$$\mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/32.$$

Proof. We know that $\mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2)$ has order 2^6 and consists of two summands. Hence it is enough to show that $\mathrm{TR}^3_{11-\lambda_4}(\mathbb{Z}; 2, \mathbb{Z}/4)$ has order 8. We do this by studying the Tate spectral sequence converging to $\pi_{*-\lambda_4}(T(\mathbb{Z})^{tC_4}; \mathbb{Z}/4)$ and restricting to the second quadrant.

We start by computing $\mathrm{TR}^2_{11-\lambda_2}(\mathbb{Z};2) \cong \mathbb{Z}/8$, which means that

$$\pi_{11-\lambda_4}(T(\mathbb{Z})^{tC_4};\mathbb{Z}/4) \cong \mathbb{Z}/4.$$

When restricting to the second quadrant (which in this case means filtration less than or equal to 8) we pick up one extra class for each differential entering the second quadrant. The only possible classes that can support such differentials with target in total degree 11 are t^{-6}, $2t^{-6}$ and $t^{-5}u_2\lambda_1$.

There is a differential $d_4(t^{-7}u_2) = t^{-5}u_2\lambda_1$, so $t^{-5}u_2\lambda_1$ does not give us an extra class. (In the corresponding Tate spectral sequence with integral coefficients the class $t^{-7}u_2$ does not exist, and $t^{-5}u_2\lambda_1$ supports a differential giving an extra integral class.) We also know that there is a differential $d_{12}(t^{-6}) = \lambda_1\mu_1^2$. Hence it suffices to show that $2t^{-6}$ is a permanent cycle. (Again, $2t^{-6}$ supports a longer differential in the corresponding Tate spectral sequence with integral coefficients.)

The C_2-Tate spectral sequence with mod 4 coefficients has been worked out by Rognes, and is (mostly) described in [8, Fig. 4.3]. In particular t^{-4} is a permanent cycle. The $RO(S^1)$-graded C_2-Tate spectral sequence is a shifted copy of the integral one, and it follows that t^{-6} is a permanent cycle in the spectral sequence converging to $\pi_{*-\lambda_4}(T(\mathbb{Z})^{tC_2};\mathbb{Z}/4)$.

Next we use that the Verschiebung (or transfer) map $V : T(\mathbb{Z})^{tC_2} \to T(\mathbb{Z})^{tC_4}$ gives a map of spectral sequences, and that $V(t^{-6}) = 2t^{-6}$. It follows that $2t^{-6}$ is a permanent cycle in the spectral sequence converging to $\pi_{*-\lambda_4}(T(\mathbb{Z})^{tC_4};\mathbb{Z}/4)$, which was what we needed to show.

$\square$

We also conjecture the computation of two other K-theory groups.

Conjecture 4.3. *Let $A = \mathbb{Z}[x,y]/(xy)$ and let $I = (x,y)$. Then*

$$\begin{aligned} K_9(A,I) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \\ K_{13}(A,I) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/5 \end{aligned}$$

Sketch proof. This conjecture is based on a conjectural understanding of the C_4-Tate spectral sequence with integral coefficients. We compute that

$$K_9(A,I) \cong \mathrm{TR}^3_{9-\lambda_4}(\mathbb{Z};2) \oplus \mathrm{TR}^1_{9-\lambda_3}(\mathbb{Z};2) \oplus \mathrm{TR}^2_{9-\lambda_3}(\mathbb{Z};3) \oplus \mathrm{TR}^1_{9-\lambda_2}(\mathbb{Z};3).$$

We find that $\mathrm{TR}^1_{9-\lambda_3}(\mathbb{Z};2) \cong \mathbb{Z}/2$, $\mathrm{TR}^2_{9-\lambda_3}(\mathbb{Z};3) \cong \mathbb{Z}/3$ and $\mathrm{TR}^1_{9-\lambda_2}(\mathbb{Z};3) \cong \mathbb{Z}/3$. The final group, $\mathrm{TR}^3_{9-\lambda_4}(\mathbb{Z};2)$, is conjecturally computed in Conjecture 4.4 below.

Similarly we compute

$$\begin{aligned} K_{13}(A,I) \cong{} & \mathrm{TR}^3_{13-\lambda_4}(\mathbb{Z};2) \oplus \mathrm{TR}^2_{13-\lambda_6}(\mathbb{Z};2) \oplus \mathrm{TR}^1_{13-\lambda_5}(\mathbb{Z};2) \oplus \mathrm{TR}^2_{13-\lambda_3}(\mathbb{Z};3) \\ & \oplus \mathrm{TR}^2_{13-\lambda_6}(\mathbb{Z};3) \oplus \mathrm{TR}^1_{13-\lambda_4}(\mathbb{Z};3) \oplus \mathrm{TR}^2_{13-\lambda_5}(\mathbb{Z};5) \oplus \mathrm{TR}^1_{13-\lambda_2}(\mathbb{Z};5). \end{aligned}$$

We conjecture the group structure of $\mathrm{TR}^3_{13-\lambda_4}(\mathbb{Z};2)$ in Conjecture 4.5 below, and the other groups are readily computed. $\square$

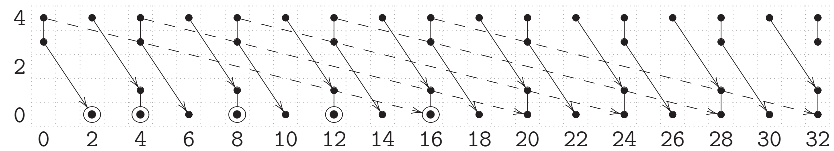

Figure 1: Degree 9 and 10 of the Tate spectral sequence converging to $\pi_{*-\lambda_4} T(\mathbb{Z})^{tC_4}$

Conjecture 4.4. *We have*

$$\mathrm{TR}^3_{9-\lambda_4}(\mathbb{Z};2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/16.$$

Sketch proof. We know that $\mathrm{TR}^3_{9-\lambda_4}(\mathbb{Z};2)$ has order 2^5 and consists of two summands. To compute this term we study the Tate spectral sequence converging to $\pi_{*-\lambda_4} T(\mathbb{Z})^{tC_4}$ and restrict to the second quadrant. (Which, again, means filtration less than or equal to 8.) We start by computing $\mathrm{TR}^2_{9-\lambda_2}(\mathbb{Z};2) \cong \mathbb{Z}/8$. Now we conjecture that the C_4-Tate spectral sequence in total degree 9 and 10 behaves as in Figure 1. The d_4 and d_5 differentials follow by comparing with the C_2-Tate spectral sequence, but we have not been able to rule out that the longer differentials could behave in a more complicated way. In the Tate spectral sequence converging to $\pi_{9-\lambda_4} T(\mathbb{Z})^{tC_4} \cong \mathbb{Z}/8$, the following classes should survive:

$$\{t^{-1}\lambda_1\mu_1, t^3\lambda_1\mu_1^3, t^7\lambda_1\mu_1^5\}.$$

These are then connected by hidden multiplication by 2 extensions.

When restricting to the second quadrant we pick up two more classes, coming from the differentials originating from t^{-5} and $2t^{-5}$. so we would have the following classes, circled in Figure 1:

$$\{t^{-3}\lambda_1, t^{-1}\lambda_1\mu_1, t^3\lambda_1\mu_1^3, t^7\lambda_1\mu_1^5, t^{11}\lambda_1\mu_1^7\}.$$

Now consider the class $t^4 u_2 \lambda_1 \mu_1^4$. This class kills $2t^7\lambda_1\mu_1^5$, while in the corresponding spectral sequence with mod 2 coefficients it kills $t^{11}\lambda_1\mu_1^7$. This implies that we have a hidden multiplication by 2 extension connecting the classes $t^7\lambda_1\mu_1^5$ and $t^{11}\lambda_1\mu_1^7$. Hence the class $t^{-1}\lambda_1\mu_1$ has order 16 and $\mathrm{TR}^3_{9-\lambda_4}(\mathbb{Z};2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/16$.

□

Conjecture 4.5. *We have*

$$\mathrm{TR}^3_{13-\lambda_4}(\mathbb{Z};2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8.$$

Sketch proof. The argument is similar to the one for Conjecture 4.4. We know that $\mathrm{TR}^3_{13-\lambda_4}(\mathbb{Z};2)$ has order 2^4 and consists of two summands, so the group is either $\mathbb{Z}/4 \oplus \mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/8$. We find that $\mathrm{TR}^2_{13-\lambda_2}(\mathbb{Z};2) \cong \mathbb{Z}/4$. In

the Tate spectral sequence converging to $\pi_{13-\lambda_4} T(\mathbb{Z})^{tC_4} \cong \mathbb{Z}/4$, the following classes should survive:

$$\{t^{-3}\lambda_1\mu_1, t\lambda_1\mu_1^3\}.$$

Again these are connected by a hidden multiplication by 2 extension.

When restricting to the second quadrant we pick up two more classes, coming from the differentials originating from $2t^{-7}$ and $t^{-6}\lambda_1 u_2$. Conjecturally we have $d_{24}(2t^{-7}) = t^5\lambda_1\mu_1^5$, so one of the new classes is $t^5\lambda_1\mu_1^5$. Now consider the class $t^{-2}\lambda_1\mu_1^2 u_2$. There is a differential $d_5(t^{-2}\lambda_1\mu_1^2 u_2) = 2t\lambda_1\mu_1^3$, while in the corresponding spectral sequence with mod 2 coefficients we have $d_{13}(t^{-3}\lambda_1\mu_1^2 u_2) = t^5\lambda_1\mu_1^5$.

As in the sketch proof of Conjecture 4.4 this implies that the classes $t\lambda_1\mu_1^3$ and $t^5\lambda_1\mu_1^5$ are connected by a hidden multiplication by 2 extension, so we have a copy of $\mathbb{Z}/8$ generated by $t^{-3}\lambda_1\mu_1$.

□

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